

# TWISTED CONVOLUTION AND MOYAL STAR PRODUCT OF GENERALIZED FUNCTIONS

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**ABSTRACT.** We consider nuclear function spaces on which the Weyl-Heisenberg group acts continuously and study the basic properties of the twisted convolution product of the functions with the dual space elements. The final theorem characterizes the corresponding algebra of convolution multipliers and shows that it contains all sufficiently rapidly decreasing functionals in the dual space. Consequently, we obtain a general description of the Moyal multiplier algebra of the Fourier-transformed space. The results extend the Weyl symbol calculus beyond the traditional framework of tempered distributions.

## 1. INTRODUCTION

The twisted convolution product of functions  $g_1(s)$  and  $g_2(s)$  on a linear symplectic space is a noncommutative deformation of the ordinary convolution and is defined by the formula

$$(g_1 \circledast_\theta g_2)(s) = \int g_1(t) g_2(s - t) e^{\frac{i}{2}\theta(s,t)} dt, \quad (1)$$

where  $\theta$  is the bilinear skew-symmetric form specifying the symplectic structure.<sup>1</sup> The Fourier transform converts the twisted convolution into the Weyl-Groenewold-Moyal star product  $\star_\theta$  which gives the composition rule for the Weyl symbols of quantum mechanical operators and plays a key role in the Weyl quantization, see [1], [2]. It is worth noting that the composition rule for phase-space functions, which corresponds to the composition of operators on a Hilbert space, was originally written by von Neumann [3] just in terms of the twisted convolution. It is customary to define the star multiplication and twisted convolution first for smooth and rapidly decreasing functions in the Schwartz space  $S$ , which forms an associative topological algebra under either of the two operations. But in practice, depending on the problem under study, we must consider an extension of these operations to one or another subspace of the dual space  $S'$  of tempered distributions. Antonets proposed a maximal extension by duality that consisted in constructing the multiplier algebra of the algebra  $(S, \star_\theta)$  [4]-[6] or, equivalently, of the algebra  $(S, \circledast_\theta)$ . This extension was later studied in many papers and most thoroughly in [7]-[10] (a detailed review and references can be found in [11]).

Field theory models on noncommutative spaces based on using the Moyal star product have been intensively investigated in the last 15 years, (see, e.g., [12] for an introduction to this topic). The interest in noncommutative spaces was stimulated by the in-depth analysis of the quantum limitations on the accuracy of localization of

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<sup>1</sup>The twisted convolution operation is often denoted by  $\ast_\theta$ , but we use the symbol  $\circledast_\theta$  here to avoid confusion with the star product  $\star_\theta$ .

space-time events in quantum theory including gravity [13], [14] and by the study of the low energy limit of string theory [15]. There is reason [16]-[18] to believe that the framework of tempered distributions is too narrow for a consistent formulation of the general principles of noncommutative quantum field theory. The Moyal product is non-local and its expansion in powers of the noncommutativity parameter  $\theta$  converges only on analytic test functions whose Fourier transforms decrease at infinity faster than the Gaussian exponential function [19], [20]. An analysis of microcausality violations in the simplest noncommutative models [21]-[23] indicates a possible connection between noncommutative field theory and the previously considered nonlocal theories, which treat quantum fields as operator-valued generalized functions defined on a suitable space of analytic test functions instead of the Schwartz space. The problem of appropriately generalizing the Weyl symbol calculus arises.

In [19], we established a condition under which a nuclear test function space  $E(\mathbb{R}^d)$  with the structure of a topological algebra under ordinary convolution is also an algebra under the twisted convolution and its Fourier-conjugate space is hence an algebra under the Moyal product. This condition can be written as

$$e^{-\frac{i}{2}\theta} \in M(E \widetilde{\otimes} E), \quad (2)$$

where  $E \widetilde{\otimes} E$  is the completed projective tensor product identifiable with  $E(\mathbb{R}^{2d})$  by the kernel theorem and  $M(E \widetilde{\otimes} E)$  is the space of its multipliers with respect to ordinary pointwise multiplication. In the case of Gel'fand-Shilov spaces  $S_\beta^\alpha$  [24] considered in [19], condition (2) leads to the restriction  $\alpha \geq \beta$  on the specifying indices. Palamodov precisely described the set of pointwise multipliers for the spaces  $S_\beta^\alpha$  [25], and their corresponding algebras of Moyal multipliers were constructed in [26].

Here, we show that the problem under discussion can be solved in a general form and a Moyal multiplier algebra can be constructed for any complete, nuclear, barrelled<sup>2</sup> space on which the Weyl-Heisenberg group acts continuously. We also describe the elements of these algebras using only well-known facts from the theory of tensor products of nuclear spaces [27], which allows avoiding complicated analytic estimates. The basic observation is that condition (2) implies that

$$v \circledast_\theta g \in M(E) \quad \text{and} \quad g \circledast_\theta v \in M(E) \quad \text{for all } g \in E, v \in E', \quad (3)$$

where  $M(E)$  is the space of pointwise multipliers for  $E$ . This allows characterizing those elements of the dual space  $E'$  that are multipliers of the algebra  $(E, \circledast_\theta)$  sufficiently exactly.

The paper is organized as follows. In Sec. 2, we consider function spaces endowed with a continuous linear action of the Weyl-Heisenberg group and define the twisted convolution of elements of the space with elements of its dual. We also present the noncommutative analogues of some properties of the usual convolution. In Sec. 3 we introduce the basic notion of a twisted convolution multiplier. Section 4 contains the main theorems; there, we show that the implication  $(2) \Rightarrow (3)$  holds under natural assumptions on the spaces under consideration, and using this implication, we obtain a characterization of the algebra of twisted convolution multipliers. Section 5 is devoted to the most interesting case of spaces invariant under the Fourier transform. In Sec. 6 we illustrate the general construction with concrete examples. The appendix contains the proof of a useful simple criterion for the continuity of the action of topological groups on spaces of the considered type.

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<sup>2</sup>Practically all spaces used in the theory of generalized functions have these properties, see [27] for their definition and role in functional analysis.

## 2. THE WEYL-HEISENBERG GROUP AND THE TWISTED CONVOLUTION

Let  $E$  be a locally convex space of complex-valued functions on  $\mathbb{R}^d$  and let  $\theta$  be a bilinear symplectic form, i.e., a nondegenerate skew-symmetric inner product on  $\mathbb{R}^d$ . We assume that the twisted shift operator

$$\tau_{\theta,s}: g(t) \rightarrow e^{\frac{i}{2}\theta(s,t)}g(t-s), \quad g \in E, \quad (4)$$

is defined and continuous on  $E$  for each  $s \in \mathbb{R}^d$ . The operators  $\tau_{\theta,s}$  realize a projective representation of the translation group to which corresponds a linear representation of its central extension generated by the initial symplectic structure, i.e. of the Weyl-Heisenberg group consisting of elements of the form  $a = (\alpha, s)$ , where  $\alpha \in \mathbb{R}$  and  $s \in \mathbb{R}^d$ , with the multiplication law

$$a_1 a_2 = \left( \alpha_1 + \alpha_2 + \frac{1}{2}\theta(s_1, s_2), s_1 + s_2 \right).$$

We also assume that  $E$  is invariant under the complex conjugation  $g \rightarrow g^*$ . Then, along with the representation  $a \rightarrow e^{i\alpha}\tau_{\theta,s}$ , the conjugate representation  $a \rightarrow e^{-i\alpha}\bar{\tau}_{\theta,s}$  is realized on  $E$  with

$$\bar{\tau}_{\theta,s}: g(t) \rightarrow e^{-\frac{i}{2}\theta(s,t)}g(t-s). \quad (5)$$

If  $E$  is also invariant under the coordinate reflection  $g(t) \rightarrow \check{g}(t) := g(-t)$ , then the functions

$$(v \circledast_{\theta} g)(s) := \left\langle v, e^{\frac{i}{2}\theta(s,\cdot)}g(s-\cdot) \right\rangle, \quad (g \circledast_{\theta} v)(s) := \left\langle v, e^{-\frac{i}{2}\theta(s,\cdot)}g(s-\cdot) \right\rangle, \quad (6)$$

are well defined for any  $g \in E$  and any  $v \in E'$ . We call them the twisted convolution products of the function  $g$  with the functional  $v$ .

*Remark 1.* The symplectic form entering in the definition of the Weyl-Heisenberg group is usually assumed to be standard and determined by the matrix  $J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$ , where  $2n = d$ . But for any other form  $\theta$ , the group is obviously the same up to an isomorphism because every symplectic space has a symplectic basis [28]. In the Wigner-Weyl representation, or phase-space formulation of quantum mechanics, the twisted convolution is determined by the matrix  $\hbar J$ , where the Planck constant  $\hbar$  plays the role of a noncommutative deformation parameter.

Although operators (4) and (5) and also functions (6) carry the subscript  $\theta$ , we omit it in what follows if it cannot cause confusion. We also assume that  $E$  is a dense vector subspace of the Schwartz space  $S$ , although definition (6) is meaningful even in a more general situation, for example, when  $E$  is continuously and densely embedded into  $L^2$  and there is hence a canonical embedding  $E \rightarrow E'$  and formulas (6) extend the initial twisted convolution operation on the elements of  $E$ . As usual, we let  $C(\mathbb{R}^d)$  denote the space of all continuous functions on  $\mathbb{R}^d$  and give it the topology of uniform convergence on compact sets.

**Proposition 1.** *Let  $E$  be a locally convex space of complex-valued functions on  $\mathbb{R}^d$ . If the reflection transformation  $g \rightarrow \check{g}$  and operators (4) and (5) act continuously on  $E$ , then functions (6) belong to  $C(\mathbb{R}^d)$ . If, in addition,  $E$  is barrelled, then the maps  $(g, v) \rightarrow v \circledast g$  and  $(g, v) \rightarrow g \circledast v$  from  $E \times E'$  to  $C(\mathbb{R}^d)$  are separately continuous.*

*Proof.* Formulas (6) can be rewritten as

$$(v \circledast)(s) = \langle v, \tau_s \check{g} \rangle, \quad (g \circledast v)(s) = \langle v, \bar{\tau}_s \check{g} \rangle.$$

The continuity of the action of  $\tau_s$  on  $E$  means that the map  $\mathbb{R}^d \times E \rightarrow E: (s, g) \rightarrow \tau_s g$  is continuous. Hence, functions (6) are obviously continuous in  $s$  under our assumptions. If  $g$  is fixed and  $s$  ranges over a compact set  $K$  in  $\mathbb{R}^d$ , then  $\tau_s \check{g}$  ranges over a compact set  $Q$  in  $E$ . Therefore, the map  $v \rightarrow v \circledast g$  is continuous in the strong topology of  $E'$  by its definition as the topology of uniform convergence on bounded sets. Analogously, the map  $v \rightarrow g \circledast v$  is continuous. Now let  $v$  be fixed. The set of operators  $\tau_s$ ,  $s \in K$ , is pointwise bounded; hence, if  $E$  is barrelled, then this set is equicontinuous by the general Banach-Steinhaus theorem (Sec. III.4.2 in [27]). Therefore, for any neighborhood  $U$  of the origin in  $E$ , there exists a neighborhood  $V$  such that  $\tau_s \check{g} \in U$  for all  $g \in V$  and for any  $s \in K$ . Taking  $U$  to be a neighborhood on which  $v$  is bounded by a number  $\epsilon > 0$ , we conclude that the map  $g \rightarrow v \circledast g$  is continuous. It can be similarly verified that the map  $g \rightarrow g \circledast v$  is continuous. The proposition is proved.  $\square$

*Remark 2.* Our main theorems concern complete nuclear spaces. Such spaces are barrelled if and only if they are reflexive and then they are Montel spaces [27]. In the appendix we prove Lemma 1, which gives easily verifiable sufficient conditions for the continuity of the action of topological transformation groups on Montel function spaces.

The operators  $\tau_s$  are automorphisms of  $C(\mathbb{R}^d)$  and it is easily seen that the linear map  $L_v: g \rightarrow v \circledast g$  commutes with them. Analogously, the map  $R_v: g \rightarrow g \circledast v$  commutes with all operators  $\bar{\tau}_s$ .

**Proposition 2.** *Let  $E$  satisfy the conditions of Proposition 1 and let  $A$  be a continuous linear map from  $E$  to  $C(\mathbb{R}^d)$ . If  $A$  commutes with all the operators  $\tau_s$  (with all  $\bar{\tau}_s$ ), then there exists a unique functional  $v \in E'$  such that  $Ag = v \circledast g$  ( $Ag = g \circledast v$ ) for all  $g \in E$ .*

*Proof.* The argument is analogous to that used for Theorem 4.2.1 in [29], where the ordinary convolution and the space  $E = C_0^\infty(\mathbb{R}^d)$  were considered. We consider the case where  $A$  commutes with all  $\tau_s$ . If such a functional  $v$  exists, then  $\langle v, \check{g} \rangle = (Ag)(0)$  for all  $g \in E$  and is hence unique. We let  $v$  denote the linear form  $g \rightarrow (A\check{g})(0)$ . Clearly,  $v \in E'$  because the operator  $A$  and the transformation  $g \rightarrow \check{g}$  are continuous. Furthermore,  $(v \circledast g)(0) = \langle v, \check{g} \rangle = (Ag)(0)$ . Using the commutativity of  $A$  with  $\tau_s$ , we obtain

$$(Ag)(-s) = (\tau_s Ag)(0) = (A\tau_s g)(0) = (v \circledast \tau_s g)(0) = (v \circledast g)(-s) \quad \text{for all } s \in \mathbb{R}^d.$$

Therefore,  $Ag = v \circledast g$  for all  $g \in E$ . The second case can be treated in the same way. The proposition is proved.  $\square$

### 3. TWISTED CONVOLUTION MULTIPLIERS

In what follows, we assume that  $E$  consists of continuous functions and the embedding  $E \rightarrow C(\mathbb{R}^d)$  is continuous. The conditions in Proposition 1 are also assumed to be satisfied. We define the spaces  $\mathcal{C}_{\theta,L}(E)$  and  $\mathcal{C}_{\theta,R}(E)$  of left and right  $\circledast$ -multipliers for  $E$  as consisting of all functionals  $v \in E'$  such that respectively  $v \circledast_\theta g \in E$  and  $g \circledast_\theta v \in E$  for each  $g \in E$  and, in addition, the maps  $g \rightarrow v \circledast_\theta g$  and  $g \rightarrow g \circledast_\theta v$  are continuous.

If the closed graph theorem is applicable to  $E$ , then the continuity requirement is satisfied automatically here and can be omitted from the definition because these maps have closed graphs. Indeed, if  $g_\gamma \rightarrow g$  and  $v \circledast_\theta g_\gamma \rightarrow h$  in  $E$ , then  $v \circledast_\theta g_\gamma \rightarrow h$  in  $C(\mathbb{R}^d)$  and  $v \circledast_\theta g = h$  by Proposition 1. It follows from Proposition 2 that there is a one-to-one correspondence between the elements of  $\mathcal{C}_L(E)$  and the continuous linear maps  $E \rightarrow E$  commuting with all  $\tau_s$ . Letting  $\mathcal{L}(E)$  denote the algebra of continuous linear operators on  $E$  equipped with the topology of uniform convergence on bounded sets, we conclude that  $\mathcal{C}_L(E)$  can be identified with its closed subalgebra. If  $v_1, v_2 \in \mathcal{C}_L(E)$ , then we

let  $v_1 \otimes v_2$  denote the element of  $\mathcal{C}_L(E)$  corresponding to the composition of operators  $g \rightarrow v_1 \otimes g$  and  $g \rightarrow v_2 \otimes g$  by Proposition 2. Explicitly, we have

$$\langle v_1 \otimes v_2, \check{g} \rangle = (v_1 \otimes (v_2 \otimes g))(0), \quad g \in E. \quad (7)$$

The natural topology of  $\mathcal{C}_L(E)$  is the topology induced by that of  $\mathcal{L}(E)$ . Analogously,  $\mathcal{C}_R(E)$  is identified with the subalgebra of operators that commute with all  $\bar{\tau}_s$ . For the elements of  $\mathcal{C}_R(E)$ , the twisted convolution product is defined by

$$\langle v_1 \otimes v_2, \check{g} \rangle = ((g \otimes v_1) \otimes v_2)(0), \quad g \in E. \quad (8)$$

If  $E$  has the structure of a topological algebra with respect to product (1), then the algebras  $\mathcal{C}_L(E)$  and  $\mathcal{C}_R(E)$  are its extensions. Both of them are unital with the Dirac  $\delta$ -function as their identity. Every  $\otimes$ -multiplier of  $E$  by duality determines the corresponding operations on the dual space  $E'$ , namely

$$\langle w \otimes v_1, g \rangle = \langle w, \check{v}_1 \otimes g \rangle, \quad \langle v_2 \otimes w, g \rangle = \langle w, g \otimes \check{v}_2 \rangle, \quad w \in E', v_1 \in \mathcal{C}_L(E), v_2 \in \mathcal{C}_R(E), \quad (9)$$

which further extends the initial operation on  $E$ .

If  $E$  is invariant under complex conjugation, then the involution  $g \rightarrow g^*$  in  $E$  induces an involution of  $E'$ . In this case  $(v \otimes g)^* = g^* \otimes v^*$  and there is hence a canonical antilinear isomorphism between  $\mathcal{C}_L(E)$  and  $\mathcal{C}_R(E)$ . We now consider the intersection

$$\mathcal{C}_\theta(E) = \mathcal{C}_{\theta,L}(E) \cap \mathcal{C}_{\theta,R}(E) \quad (10)$$

The natural topology of  $\mathcal{C}_\theta(E)$  is the least upper bound of the topologies induced by those of  $\mathcal{C}_{\theta,L}(E)$  and  $\mathcal{C}_{\theta,R}(E)$ . If  $E \cap \mathcal{C}_\theta(E)$  is dense in  $\mathcal{C}_\theta(E)$  in this topology and  $E$  consists of bounded continuous integrable functions, then  $\mathcal{C}_\theta(E)$  has the canonical structure of a unital involutive algebra with respect to the  $\otimes$ -product, because formulas (7) and (8) with  $v_1, v_2 \in \mathcal{C}_\theta(E)$  define the same functional in this case. Indeed, using the Fubini theorem, we can easily verify that under these conditions and for  $v_1, v_2 \in E$ , the integrals representing the right-hand sides of these formulas coincide, and our assertion follows by passing to the limit.

#### 4. THE MAIN THEOREMS

We must say a few words about the terminology used below. Let  $E$  be a complete nuclear locally convex space. If it is a vector subspace of another locally convex space  $E_0$  and the identical embedding  $E \rightarrow E_0$  is continuous, we say that  $E$  is a complete nuclear subspace of  $E_0$ . We need the following auxiliary statement.

**Proposition 3.** *Let  $E$  be a complete nuclear subspace of the Schwartz space  $S(\mathbb{R}^d)$ . Then  $E \tilde{\otimes} E$  is a complete nuclear subspace of  $S(\mathbb{R}^{2d})$ .*

*Proof.* We recall that the completed projective tensor product of nuclear spaces is nuclear [27]. Because  $S(\mathbb{R}^{2d}) = S(\mathbb{R}^d) \tilde{\otimes} S(\mathbb{R}^d)$ , there is a natural continuous linear map  $\iota: E \otimes E \rightarrow S(\mathbb{R}^{2d})$ , and we need only show that its extension  $\tilde{\iota}$  by continuity to  $E \tilde{\otimes} E$  is injective. If  $E$  is complete and nuclear, then by the Grothendieck theorem (Sec. IV.9.4 in [27]) the space  $E \tilde{\otimes} E$  is identified with the space  $\mathcal{B}_e(E'_\sigma, E'_\sigma)$  of separately continuous bilinear forms on  $E'_\sigma \times E'_\sigma$ , where the subscripts  $\sigma$  and  $e$  respectively mean that  $E'$  is equipped with the weak topology and that  $\mathcal{B}$  is equipped with the topology of biquicontinuous convergence. The canonical map  $E \times E \rightarrow \mathcal{B}_e(E'_\sigma, E'_\sigma)$  takes each pair of functions  $(f, g)$  to the bilinear form

$$(f \otimes g)(u, v) = \langle u, f \rangle \langle v, g \rangle, \quad u, v \in E'. \quad (11)$$

We let  $j'$  denote the transpose of the embedding  $j: E \rightarrow S(\mathbb{R}^d)$ . Applying the above-noted identification also to the Schwartz space, we see that  $\tilde{\iota}$  maps the bilinear form  $\mathbf{b} \in \mathcal{B}_e(E'_\sigma, E'_\sigma)$  to the bilinear form on  $S' \times S'$  whose value on a pair of distributions

$u, v \in S'$  is  $\mathbf{b}(j'(u), j'(v))$ . Because  $j$  is injective,  $j'(S')$  is dense in  $E'_\sigma$  and  $\tilde{\iota}(\mathbf{b}) = 0$  hence implies  $\mathbf{b} = 0$ , which completes the proof.  $\square$

If the conditions of Proposition 3 are satisfied, then we use  $E(\mathbb{R}^{2d})$  and  $E \tilde{\otimes} E$  equivalently to denote the same space. A simple proof of the kernel theorem generalizing the famous Schwartz theorem concerning  $S(\mathbb{R}^d)$  to any complete nuclear barrelled space is given in [30]. It develops a construction proposed by Grothendieck (Theorem 13 in Chap. 2 in [31]). In practice,  $E(\mathbb{R}^{2d})$  is determined by the same restrictions as  $E(\mathbb{R}^d)$  but imposed on functions of the doubled number of variables, see examples in Sec. 6.

It is well known that the (ordinary) convolution of any tempered distribution with any test function in the Schwartz space  $S$  is a multiplier of this space with respect to pointwise multiplication. The next theorem establishes an analogue of this property for larger classes of generalized functions and shows that it is preserved under the noncommutative deformation of convolution. For any locally convex space  $E \subset S$ , a continuous function  $\mu$  is called a pointwise multiplier of  $E$  if  $\mu g \in E$  for all  $g \in E$  and the map  $g \rightarrow \mu g$  is continuous.<sup>3</sup> We let  $M(E)$  denote the set of all such multipliers and give it the topology induced by that of the operator algebra  $\mathcal{L}(E)$ .

**Theorem 1.** *Let  $E$  be a complete nuclear barrelled subspace of  $S(\mathbb{R}^d)$  and let  $\theta(s, t)$  be a symplectic form on  $\mathbb{R}^d$ . If  $e^{-\frac{i}{2}\theta} \in M(E \tilde{\otimes} E)$  and the involutive transformation  $h(s, t) \rightarrow h(s, s - t)$  is a continuous automorphism of  $E(\mathbb{R}^{2d}) = E \tilde{\otimes} E$ , then  $E$  is an algebra under the  $\otimes_\theta$ -product, this product is separately continuous, the functions  $v \otimes_\theta g$  and  $g \otimes_\theta v$  are well defined and belong to  $M(E)$  for any  $g \in E$  and any  $v \in E'$ , and the maps  $(g, v) \rightarrow v \otimes_\theta g$  and  $(g, v) \rightarrow g \otimes_\theta v$  from  $E \times E'$  into  $M(E)$  are also separately continuous.*

*Proof.* We first show that if  $h \in E(\mathbb{R}^{2d})$  and  $s$  is fixed, then  $h(s, t)$  regarded as a function of the variable  $t$  belongs to  $E$ . As in the proof of Proposition 3, we identify  $h$  with a bilinear separately continuous form  $\mathbf{h}$  on  $E'_\sigma \times E'_\sigma$ . We note that

$$h(s, t) = \mathbf{h}(\delta_s, \delta_t), \quad (12)$$

where the functional  $\delta_s \in E'$  is defined by  $\langle \delta_s, f \rangle = f(s)$ . Indeed, (11) implies (12) for all linear combinations of functions of the form  $h = f \otimes g$  and (12) holds by continuity for any element of  $E(\mathbb{R}^{2d})$ . The linear map  $E' \rightarrow \mathbb{C}: v \rightarrow \mathbf{h}(\delta_s, v)$  is continuous in the weak topology  $\sigma(E', E)$ . Therefore, there exists a function  $h_s \in E = (E'_\sigma)'$  such that  $\langle v, h_s \rangle = \mathbf{h}(\delta_s, v)$  for all  $v \in E'$ . Substituting  $v = \delta_t$  here, we obtain  $h_s(t) = h(s, t)$ . Clearly, the map  $h \rightarrow h_s$  is continuous if  $E(\mathbb{R}^{2d})$  and  $E$  are endowed with the weak topologies. Furthermore, it is easy to see that the function  $s \rightarrow \langle v, h(s, \cdot) \rangle = \mathbf{h}(\delta_s, v)$  belongs to  $E$ . Indeed, the linear map  $u \rightarrow \mathbf{h}(u, v)$  is continuous in the topology  $\sigma(E', E)$  and there hence exists a function  $h_v \in E$  such that  $\langle u, h_v \rangle = \mathbf{h}(u, v)$  for all  $u \in E'$ . For  $u = \delta_s$ , this equality becomes  $h_v(s) = \langle v, h(s, \cdot) \rangle$ . The map  $E' \rightarrow E: v \rightarrow h_v$  is also continuous in the weak topologies. Moreover,  $h_v$  depends weakly continuously on  $h$ .

We apply the above consideration to the function  $h(s, t) = f(s)g(s - t)$ , where  $f, g \in E$ . By the conditions in the theorem, it belongs to  $E(\mathbb{R}^{2d})$ . We fix a point  $s \in \mathbb{R}^d$  and choose  $f$  such that  $f(s) = 1$ . By what was said above, the function  $t \rightarrow g(s - t)$  belongs to  $E$  and the map  $E \rightarrow E: g(t) \rightarrow g(s - t)$  is weakly continuous. Every barrelled space is a Mackey space, it hence follows that this map is continuous (Sec. IV.7.4 in [27]). Because  $s$  can be chosen arbitrarily, we see that the coordinate reflection and translations are continuous operators on  $E$ . Similar reasoning applied to  $f(s)e^{-\frac{i}{2}\theta(s, t)}g(s - t)$  shows that the function  $t \rightarrow e^{-\frac{i}{2}\theta(s, t)}g(s - t)$  belongs to  $E$  and the

<sup>3</sup>The continuity condition is automatically satisfied if the closed graph theorem is applicable to  $E$ .

map  $g(t) \rightarrow e^{-\frac{i}{2}\theta(s,t)}g(s-t)$  is continuous. Therefore, twisted shifts (4) and (5) are also continuous operators on  $E$ , and the definitions of  $v \circledast g$  and  $g \circledast v$  in Sec. 2 are applicable here. Furthermore, because  $f(s)\langle v, e^{\pm \frac{i}{2}\theta(s,\cdot)}g(s-\cdot) \rangle \in E$  for any  $f \in E$ , the functions  $\langle v, \tau_s \check{g} \rangle$  and  $\langle v, \bar{\tau}_s \check{g} \rangle$  are continuous in  $s$  for each  $v \in E'$ . Hence, the sets  $\{\tau_s \check{g}: |s| \leq 1\}$  and  $\{\bar{\tau}_s \check{g}: |s| \leq 1\}$ , where  $g$  is fixed, are weakly bounded. These sets are then bounded in the original topology of  $E$  (Sec. IV.3.2 in [27]) and the corresponding representations of the Weyl-Heisenberg group in  $E$  are continuous by Lemma 1 proved in the appendix. For any fixed  $v$  and  $g$ , the functions  $f \cdot (v \circledast g)$  and  $f \cdot (g \circledast v)$  are weakly continuous in  $f$ , and the multiplication by  $v \circledast g$  and by  $g \circledast v$  are therefore continuous operators from  $E$  to  $E$ . We conclude that the twisted convolution products belong to  $M(E)$ . Furthermore, again by what was said above, the function  $t \rightarrow \langle u, f(\cdot)e^{-\frac{i}{2}\theta(\cdot,t)}g(t-\cdot) \rangle$  belongs to  $E$  for any  $u \in E'$ . In particular,

$$\int f(s)e^{-\frac{i}{2}\theta(s,t)}g(t-s)ds = f \circledast g \in E.$$

Therefore,  $E$  is an algebra under the twisted convolution. Clearly, the maps  $f \rightarrow f \circledast g$  and  $g \rightarrow f \circledast g$  are weakly continuous and are hence continuous in the original topology of  $E$ .

It remains to prove that the maps  $(g, v) \rightarrow v \circledast g$  and  $(g, v) \rightarrow g \circledast v$  from  $E \times E'$  to  $M(E)$  are separately continuous. Let  $v$  be fixed. We show that for each neighborhood  $U$  of the origin and for each bounded set  $Q$  in  $E$ , there exists a neighborhood  $W$  of the origin such that  $f \cdot (v \circledast g) \in U$  for all  $f \in Q$  and for all  $g \in W$ . By the definition of the topology of bieuicontinuous convergence, a base of neighborhoods of the origin in  $E \widetilde{\otimes} E = \mathcal{B}_e(E'_\sigma, E'_\sigma)$  is formed by sets of the form

$$\mathcal{U}_{U,V} = \left\{ \mathbf{b} \in \mathcal{B}(E'_\sigma, E'_\sigma): \sup_{u \in U^\circ, v \in V^\circ} |\mathbf{b}(u,v)| \leq 1 \right\}, \quad (13)$$

where  $U$  and  $V$  run over a basis of absolutely convex closed neighborhoods of zero in  $E$ , and  $U^\circ, V^\circ$  are their polars, i.e.,  $U^\circ = \{u \in E': \sup_{f \in U} |\langle u, f \rangle| \leq 1\}$ . The functional  $v$  is bounded by unity on a zero-neighborhood which we take as  $V$  in (13). By the conditions of the theorem, for any  $\mathcal{U}_{U,V}$ , we can find neighborhoods  $U_1$  and  $V_1$  such that

$$h(s, t) = f(s)e^{\frac{i}{2}\theta(s,t)}g(s-t) \in \mathcal{U}_{U,V}$$

for all  $f \in U_1$  and  $g \in V_1$ . For the bounded set  $Q$ , there exists  $\delta > 0$  such that  $Q \subset \delta U_1$ . We set  $W = \delta V_1$ . Then for all  $u \in U^\circ$ ,  $f \in Q$ , and  $g \in W$ , we have

$$|\mathbf{h}(u, v)| = |\langle u, f(v \circledast g) \rangle| \leq 1.$$

In another words,  $f \cdot (v \circledast g) \in U^{\circ\circ} = U$ , which proves the continuity of the map  $E \rightarrow M(E): g \rightarrow v \circledast g$ .

Now let  $g$  be fixed and a sequence  $v_n$  tend to zero in  $E'$ . Then the sequence of multipliers  $v_n \circledast g \rightarrow 0$  in the topology of simple convergence on  $L(E)$  and hence also in the topology of bounded convergence (see Sec. III.4.6 in [27]) because  $E$  is a Montel space as noted in Remark 2. Hence, the map  $E' \rightarrow \mathcal{L}(E): v \rightarrow v \circledast g$  is sequentially continuous. It remains to show that  $E'$  is a bornological space because for such spaces the sequential continuity of a linear map into an arbitrary locally convex space implies its continuity (Sec. II.8.3 in [27]). For this, we recall that every complete nuclear space is representable as the projective limit of a suitable family of Hilbert spaces  $E_\gamma$ . The projective limit can be assumed to be reduced and the dual space  $E'$  with the Mackey topology  $\tau(E', E)$  is then the inductive limit of the family of spaces  $E'_\gamma$  equipped with the Mackey topologies  $\tau(E'_\gamma, E_\gamma)$  (Sec. IV.4.4 in [27]). Because the Montel and

Hilbert spaces are reflexive, the strong topology of their duals coincides with the Makkey topology. Therefore,  $E'$  is bornological as an inductive limit of normed spaces. We conclude the the map  $E' \rightarrow M(E): v \rightarrow v \circledast g$  is continuous. A similar argument for  $g \circledast v$  completes the proof.  $\square$

**Corollary 1.** *If the conditions of Theorem 1 are satisfied, then definition (6) is equivalent to the definition of the products  $v \circledast g$  and  $g \circledast v$  by duality, and the relations*

$$(v \circledast g) \circledast f = v \circledast (g \circledast f), \quad (g \circledast v) \circledast f = g \circledast (v \circledast f), \quad g \circledast (f \circledast v) = (g \circledast f) \circledast v. \quad (14)$$

hold for all  $v \in E'$ , and  $g, f \in E$ .

*Proof.* For any topological algebra  $(E, \circledast)$ , the twisted convolution of a function  $g \in E$  with a functional  $v \in E'$  is defined by duality by the formulas

$$\langle v \circledast g, f \rangle = \langle v, \check{g} \circledast f \rangle, \quad \langle g \circledast v, f \rangle = \langle v, f \circledast \check{g} \rangle, \quad f \in E. \quad (15)$$

The maps  $v \rightarrow v \circledast g$  and  $v \rightarrow g \circledast v$  from  $E'$  to  $E'$  are then continuous because they are the transposes of the continuous maps  $f \rightarrow \check{g} \circledast f$  and  $f \rightarrow f \circledast \check{g}$ . For  $v \in E$ , the right-hand sides of the equalities in (15) are easily seen to coincide with the result of integrating functions (6) multiplied by the test function  $f$ . We must show that this also holds for any  $v \in E'$ . For this, we note that under the conditions in Theorem 1, the space  $E$  is dense in  $E'$ . Indeed, as shown in the proof of Theorem 1, along with a function  $g(t)$ ,  $E$  contains all the shifted functions  $g(t - s)$  and also all the products  $e^{\frac{i}{2}\theta(s,t)}g(t)$ , where  $s \in \mathbb{R}^d$ . According to Sec. IV.8.4 in [24], it follows that  $E$  has sufficiently many functions in the following sense. Let a function  $\varphi$  be locally integrable. Then the condition that the integral  $\int \varphi(t)f(t)dt$  exists and is zero for all  $f \in E$  implies that  $\varphi(t) \equiv 0$ . Therefore, the canonical map  $E \rightarrow E'$  is injective, and since  $E$  is reflexive, this map has a closed range by the Hahn-Banach theorem, as stated above. Further, each multiplier  $m \in M(E)$  defines a functional  $\mu \in E'$  by the formula  $\langle \mu, f \rangle = \int m(t)f(t)dt$ , and the map

$$M(E) \rightarrow E': m \rightarrow \mu. \quad (16)$$

is injective by the same reasoning and is continuous by the definition of topology in these spaces. Approximating the functional  $v \in E'$  by functions  $v_\gamma \in E$ , using the continuity of the maps  $v \rightarrow v \circledast g$ ,  $v \rightarrow g \circledast v$  from  $E'$  to  $M(E)$ , and passing to the limit in the equality

$$\int (v_\gamma \circledast g)(t)f(t)dt = \langle v_\gamma, \check{g} \circledast f \rangle$$

and in an analogous equality for  $g \circledast v_\gamma$ , we conclude that map (16) establishes a one-to-one correspondence between the functions defined by (6) and the functionals defined by (15). Formulas (14) follow by continuity from the associativity of the algebra  $(E, \circledast)$ , which completes the proof.  $\square$

We now note that, under the conditions of Theorem 1, the space  $E$  is an algebra under pointwise multiplication. Indeed, for any  $f, g \in E$ , the function  $f(s)g(s - t)|_{t=0}$  belongs to  $E$ , and it follows from the proof of Theorem 1 that the map  $(f, g) \rightarrow f \cdot g$  is separately continuous. Hence, there is a canonical embedding  $E \rightarrow M(E)$ . We assume below that  $E$  is dense in  $M(E)$ . This condition is generally assumed for the multipliers of topological algebras and is included in the definition of  $M(E)$  in [25]. If the density condition is satisfied, then the algebra  $M(E)$  is canonically isomorphic to the closure in  $\mathcal{L}(E)$  of the set of all operators of multiplication by elements of  $E$ .

**Theorem 2.** *If the conditions of Theorem 1 are satisfied and  $E$  is dense in  $M(E)$ , then the space  $M'(E)$  dual to  $M(E)$  is contained in  $\mathcal{C}_\theta(E)$ .*



*Proof.* Let  $Q$  be a bounded set in  $E$ . The set of all continuous maps  $g \rightarrow f \cdot g$ , where  $f \in Q$ , is pointwise bounded. By the Banach-Steinhaus theorem, it follows that for each neighborhood  $U$  of the origin in  $E$ , there exists a neighborhood  $V$  such that  $f \cdot g \in U$  for all  $g \in V$  and  $f \in Q$ . This implies that the natural injection  $E \rightarrow M(E)$  is continuous. Its transpose  $M'(E) \rightarrow E'$  is hence well defined and continuous. The assumption that  $E$  is dense in  $M(E)$  implies that the latter map is injective. Therefore, if  $w \in M'(E)$ , then  $w \in E'$  and  $w \otimes g \in M(E)$  by Theorem 1. We must show that  $w \otimes g \in E$ . By Corollary 1 we have

$$(w \otimes g) \otimes f = w \otimes (g \otimes f) \quad \text{for all } g, f \in E.$$

The equality of these convolution products at zero can be written as

$$\langle w \otimes g, \check{f} \rangle = \langle \check{w}, g \otimes f \rangle. \quad (17)$$

We let  $L'_g$  denote the linear map from  $M'(E)$  to  $E$  that is transpose of the continuous map  $L_g: v \rightarrow g \otimes v$ . We then have

$$\langle v, L'_g \check{w} \rangle = \langle \check{w}, g \otimes v \rangle \quad \text{for all } v \in E'. \quad (18)$$

For  $v = f$ , the right-hand side of (17) coincides with that of (18), and the left-hand side of (18) is written as  $\int (L'_g \check{w})(t) f(t) dt$ . Because  $f$  is an arbitrary element of  $E$  and this space has sufficiently many functions, we infer that the function  $(w \otimes g)(t)$  coincides with the function  $(L'_g \check{w})(-t)$  belonging to  $E$ . It remains to show that the map  $E \rightarrow E: g \rightarrow w \otimes g$  is continuous. By Theorem 1, for any fixed  $v \in E'$ , the map  $E \rightarrow M(E): g \rightarrow g \otimes v$  is continuous, and hence *a fortiori* continuous in the weak topologies of  $E$  and  $M(E)$ . Therefore, the scalar-valued function  $g \rightarrow \langle \check{w}, L_g v \rangle$  is continuous in the topology  $\sigma(E, E')$ , and the map  $E \rightarrow E: g \rightarrow L'_g \check{w} = w \otimes g$  is hence weakly continuous. Since  $E$  is a Mackey space, this map is also continuous in its original topology. We conclude that  $w \in \mathcal{C}_L(E)$ . Analogously, we obtain  $w \in \mathcal{C}_R(E)$ . The theorem is proved.  $\square$

Theorem 2 implies a corresponding result for the space  $F$  that is Fourier-conjugate to  $E$ , i.e.,  $E = \widehat{F}$ . We let  $\hat{f}(s) := \int f(x) e^{-ix \cdot s} dx$  denote the Fourier transform of a function  $f$ . For any functions  $f_1$  and  $f_2$  in the Schwartz space  $S$ , we have the relation

$$\widehat{(f_1 \star_\theta f_2)} = (2\pi)^{-d} \hat{f}_1 \otimes_\theta \hat{f}_2, \quad (19)$$

which can be taken as the definition of the Moyal product of elements of  $S$ . Therefore, if  $E$  satisfies the conditions of Theorems 1 and 2, then  $F$  is an algebra under the  $\star_\theta$ -product with separately continuous multiplication. This product can be uniquely extended by continuity to the case where one of the factors belongs to the dual space, and if  $u \in F'$  and  $f \in F$ , then

$$\widehat{(u \star_\theta f)} = (2\pi)^{-d} \hat{u} \otimes_\theta \hat{f}, \quad \widehat{(f \star_\theta u)} = (2\pi)^{-d} \hat{f} \otimes_\theta \hat{u}.$$

Furthermore, we have the relations

$$\langle u \star_\theta g, f \rangle = \langle u, g \star_\theta f \rangle, \quad \langle g \star_\theta u, f \rangle = \langle u, f \star_\theta g \rangle, \quad f, g \in F, \quad u \in F', \quad (20)$$

which correspond to formulas (15) and extend the Moyal product to elements of  $F'$  by duality. Because  $E$  is an algebra under ordinary multiplication,  $F$  is also an algebra under ordinary convolution. We let  $C(F)$  denote the corresponding algebra of (ordinary) convolution multipliers and  $\mathcal{M}_{\theta,L}(F)$  and  $\mathcal{M}_{\theta,R}(F)$  denote the algebras of left and right Moyal multipliers for  $F$ . Then  $\widehat{C}(F) = M(\widehat{F})$ ,  $\widehat{\mathcal{M}}_{\theta,L}(F) = \mathcal{C}_{\theta,L}(\widehat{F})$ , and  $\widehat{\mathcal{M}}_{\theta,R}(F) = \mathcal{C}_{\theta,R}(\widehat{F})$ . We thus obtain the following result.

**Corollary 2.** *If  $F$  is a function space whose Fourier transform  $E = \widehat{F}$  satisfies the conditions of Theorems 1 and 2, then the space  $C'(F)$  is contained in  $\mathcal{M}_\theta(F) = \mathcal{M}_{\theta,L}(F) \cap \mathcal{M}_{\theta,R}(F)$ .*

## 5. THE CASE OF FOURIER-INVARIANT SPACES

The Fourier-invariant function spaces with the structure of an algebra under both the twisted convolution and the Moyal products are particularly interest. We recall that an autohomeomorphism of a topological space is a continuous bijection of this space onto itself, whose inverse is also continuous.

**Theorem 3.** *If a space  $E$  satisfies the conditions of Theorems 1 and 2 and, in addition, the Fourier transform and the linear changes of variables in  $\mathbb{R}^d$  are autohomeomorphisms of  $E$ , then both the spaces  $M'(E)$  and  $C'(E)$  are contained in  $\mathcal{C}_\theta(E)$  and in  $\mathcal{M}_\theta(E)$ .*

*Proof.* We can assume that the symplectic form  $\theta$  is determined by an antisymmetric matrix  $\vartheta$  via the formula  $\theta(s, t) = s \cdot \vartheta t$ , where the dot denotes the usual inner product on  $\mathbb{R}^d$ . Let  $f_1, f_2 \in E$ . It follows from (1) and (19) that

$$\begin{aligned} (f_1 \star_\vartheta f_2)(x) &= \frac{1}{(2\pi)^{2d}} \iiint \hat{f}_1(t) f_2(y) e^{\frac{i}{2}s \cdot \vartheta t - i(s-t) \cdot y + is \cdot x} dt dy ds = \\ &= \frac{1}{(2\pi)^d} \iint \hat{f}_1(t) f_2(y) e^{it \cdot y} \delta(y - x - \frac{1}{2}\vartheta t) dt dy = \frac{1}{(2\pi)^d} \int \hat{f}_1(t) f_2(x + \frac{1}{2}\vartheta t) e^{it \cdot x} dt = \\ &= \frac{1}{\pi^d \det \theta} \int \hat{f}_1(-2\theta^{-1}\xi) f_2(x - \xi) e^{-2ix \cdot \theta^{-1}\xi} d\xi. \end{aligned} \quad (21)$$

Hence, we have

$$f_1 \star_\vartheta f_2 = \frac{1}{\pi^d \det \theta} (\mathcal{F}_\vartheta f_1) \circledast_{-4\vartheta^{-1}} f_2,$$

where

$$(\mathcal{F}_\vartheta f)(\xi) := \int f(x) e^{2ix \cdot \vartheta^{-1}\xi} dx.$$

An analogous calculation gives

$$f_1 \star_\vartheta f_2 = \frac{1}{\pi^d \det \theta} f_1 \circledast_{-4\vartheta^{-1}} (\overline{\mathcal{F}_\vartheta} f_2),$$

where

$$(\overline{\mathcal{F}_\vartheta} f)(\xi) := \int f(x) e^{-2ix \cdot \vartheta^{-1}\xi} dx.$$

It follows from the conditions of the theorem that each of the two symplectic Fourier transforms  $\mathcal{F}_\vartheta$  and  $\overline{\mathcal{F}_\vartheta}$  is an autohomeomorphism of  $E$ , and, furthermore,  $e^{-2ix \cdot \vartheta^{-1}\xi} \in M(E \widetilde{\otimes} E)$ . Therefore, by Theorem 1, the twisted convolution products  $(\mathcal{F}_\vartheta f) \circledast_{-4\vartheta^{-1}} v$  and  $v \circledast_{-4\vartheta^{-1}} (\overline{\mathcal{F}_\vartheta} f)$  with the deformation parameter  $-4\vartheta^{-1}$  are well defined for all  $f \in E$  and for all  $v \in E'$ , and moreover, are continuous in  $v$ . Because the twisted convolution and the Moyal multiplication extend by continuity uniquely to the case where one of the factors belongs to the dual space, we conclude that

$$f \star_\vartheta v = \frac{1}{\pi^d \det \theta} (\mathcal{F}_\vartheta f) \circledast_{-4\vartheta^{-1}} v, \quad v \star_\vartheta f = \frac{1}{\pi^d \det \theta} v \circledast_{-4\vartheta^{-1}} (\overline{\mathcal{F}_\vartheta} f). \quad (22)$$

By Theorem 1 and formulas (22), both the products  $f \star_\vartheta v$  and  $v \star_\vartheta f$  belong to  $M(E)$ , and the maps  $(f, v) \rightarrow f \star_\vartheta v$  and  $(f, v) \rightarrow v \star_\vartheta f$  from  $E \times E'$  to  $M(E)$  are continuous. Further arguments are similar to those in the proof of Theorem 2. Let  $w \in M'(E) \subset E'$ .

Then  $w \star_{\vartheta} f \in M(E)$  by Theorem 1. We need to show that  $w \star_{\vartheta} f \in E$ . From (20), we have

$$\langle w \star_{\vartheta} f, g \rangle = \langle w, f \star_{\vartheta} g \rangle \quad \text{for all } g \in E. \quad (23)$$

Let  $L_f$  be the linear map  $v \rightarrow f \star_{\vartheta} v$  from  $E'$  to  $M(E)$ . Then

$$\langle v, L'_f w \rangle = \langle w, f \star_{\vartheta} v \rangle \quad \text{for all } v \in E'. \quad (24)$$

For  $v = g$ , the right-hand sides of equalities (23) and (24) coincide and

$$\langle g, L'_f w \rangle = \int (L'_f w)(\xi) g(\xi) d\xi.$$

Therefore,  $w \star_{\vartheta} f$  coincides with the function  $L'_f w$  belonging to  $E$ . For any fixed  $v \in E'$ , the map  $E \rightarrow M(E): f \rightarrow f \star_{\vartheta} v$  is continuous, and *a fortiori* continuous in the weak topologies of  $E$  and  $M(E)$ . The function  $f \rightarrow \langle w, L_f v \rangle$  is hence continuous in the topology  $\sigma(E, E')$ . Consequently, the map  $E \rightarrow E: f \rightarrow L'_f w = w \star_{\vartheta} f$  is weakly continuous. Because  $E$  is a Mackey space, this map is also continuous in its original topology. We conclude that  $w \in \mathcal{M}_{\theta, L}(E)$ . Analogously,  $w \in \mathcal{M}_{\theta, R}(E)$ , and hence  $M'(E) \subset \mathcal{M}_{\theta}(E)$ . From this result combined with the isomorphisms  $\widehat{M}(E) = C(\widehat{E})$  and  $\widehat{\mathcal{M}}_{\theta}(E) = \mathcal{C}_{\theta}(\widehat{E})$ , where in this case  $\widehat{E} = E$ , we deduce that  $C'(E) \subset \mathcal{C}_{\theta}(E)$ . The theorem is proved.  $\square$

*Remark 3.* It follows from formulas (22) that under the conditions of Theorem 3, the algebras  $\mathcal{M}_{\vartheta}$  and  $\mathcal{C}_{-4\vartheta-1}(E) = \widehat{\mathcal{M}}_{-4\vartheta-1}(E)$  consist of the same elements of  $E'$ . In particular, for  $\theta = \hbar J$  with  $J$  being the standard symplectic matrix, the algebras  $\mathcal{M}_{\hbar J}$  and  $\mathcal{C}_{4\hbar-1, J}(E)$  consist of the same elements, and the algebra  $\mathcal{M}_{2J}(E)$  is Fourier invariant.

## 6. EXAMPLES AND CONCLUDING REMARKS

Many of the spaces used in the theory of generalized functions are nuclear Fréchet spaces or their strong duals. They are respectively abbreviated as FN and DFN spaces. These spaces have additional nice properties. In particular, the Ptak version of the closed graph theorem is applicable to them and the separate continuity of a bilinear map of the Cartesian product of such spaces to an arbitrary locally convex space is equivalent to the joint continuity of this map. Furthermore, the completed projective tensor product of two FN or DFN spaces is respectively an FN space or a DFN space (see. [27], [32]). We note that every FN space is also an FS (Fréchet-Schwartz) space and every DFN space is a DFS space. A survey of basic properties of FS and DFS spaces can be found, for example, in [33], [34]. The formulations and proofs given above can be considerably simplified for these spaces. The Schwartz space  $S$  is the best-known example of an FN space. The spaces  $\mathcal{S}^{\beta}$  considered in [22], [35] in the context of noncommutative field theory also belong to this class. The space  $\mathcal{S}^{\beta}(\mathbb{R}^d)$ ,  $\beta > 0$ , is defined as the projective limit  $\text{projlim}_{N \rightarrow \infty, B \rightarrow 0} S_N^{\beta, B}(\mathbb{R}^d)$ , where  $S_N^{\beta, B}(\mathbb{R}^d)$  is the

Banach space of smooth functions on  $\mathbb{R}^d$  with the finite norm

$$\|f\|_{N, B} = \sup_{x, \kappa} (1 + |x|)^N \frac{|\partial^{\kappa} f(x)|}{B^{|\kappa|} \kappa^{\beta \kappa}}. \quad (25)$$

(Here and hereafter, we use the standard multi-index notation adopted in [24], [29].) The Fourier transformed space  $\widehat{\mathcal{S}}^{\beta} = \mathcal{S}_{\beta}$  belongs to the class of spaces  $K(M_n)$ , where in this case  $M_n(p) = e^{n|p|^{1/\beta}}$ , and satisfies the nuclearity condition  $\int (M_n/M_{n'}) dp < \infty$ ,  $n' > n$ , indicated in [24]. If  $\beta > 1$ , then the space  $\mathcal{S}^{\beta}$  contains functions of compact support, and the elements of its dual  $(\mathcal{S}^{\beta})'$  are said to be tempered ultradistributions

(of the Beurling type). For  $\beta \leq 1$ , the functions in  $\mathcal{S}^\beta(\mathbb{R}^d)$  allow analytic continuation to  $\mathbb{C}^d$  as entire functions of order  $\leq 1/(1 - \beta)$ . The space  $\mathcal{S}^1$  plays a particular role. Following Morimoto [36], we call the elements of  $(\mathcal{S}^1)'$  tempered ultrahyperfunctions; they were used in axiomatic formulation of nonlocal quantum field theory with a fundamental length (see [30], [37], [38] and references therein). It is easy to see that  $e^{-\frac{i}{2}\theta}$  is a pointwise multiplier of  $\mathcal{S}_\beta(\mathbb{R}^{2d})$  for any real bilinear form  $\theta$  on  $\mathbb{R}^d$ . The space  $\mathcal{S}_\beta$  thus satisfies all the conditions of Theorem 1 and is hence an algebra under the star product  $\star_\theta$  for any  $\beta > 0$ . In the study of noncommutative quantum field theory models, it is common practice to represent the Moyal product as the power series

$$(f \star_\theta g)(x) = f(x)g(x) + \sum_{n=1}^{\infty} \left(\frac{i}{2}\right)^n \frac{1}{n!} \vartheta^{\mu_1 \nu_1} \dots \vartheta^{\mu_n \nu_n} \partial_{\mu_1} \dots \partial_{\mu_n} f(x) \partial_{\nu_1} \dots \partial_{\nu_n} g(x), \quad (26)$$

where  $\vartheta^{\mu\nu}$  is the matrix of the symplectic form  $\theta$ . It was shown in [22] that if  $\beta \leq 1/2$ , then series (26) converges absolutely for all  $f, g \in \mathcal{S}^\beta$  in each of norms (25). The space  $\mathcal{S}^{1/2}$  is a maximal FN subspace of  $S$  for which representation (26) is absolutely convergent, and this space hence plays a special role in noncommutative field theory. Let  $E_N^{\beta, B}(\mathbb{R}^d)$  be the Banach space of smooth functions on  $\mathbb{R}^d$  with the finite norm

$$\|f\|_{-N, B} = \sup_{x, \kappa} (1 + |x|)^{-N} \frac{|\partial^\kappa f(x)|}{B^{|\kappa|} \kappa^{\beta \kappa}} \quad (27)$$

and let  $\mathcal{E}^\beta = \text{inj} \lim_{N \rightarrow \infty} \text{proj} \lim_{B \rightarrow 0} E_N^{\beta, B}$ .

**Theorem 4.** *The space  $\mathcal{E}^\beta$  is contained in  $\mathcal{M}_\theta(\mathcal{S}^\beta)$ . For each  $w \in \mathcal{E}^\beta$  and each  $v \in (\mathcal{S}^\beta)'$ , the Moyal products  $w \star_\theta v$  and  $v \star_\theta w$  are well defined as elements of  $(\mathcal{S}^\beta)'$ .*

*Proof.* By Corollary 2, to prove the first statement, it suffices to show the inclusion  $\mathcal{E}^\beta \subset C'(\mathcal{S}^\beta)$ . Then the second statement also holds because the products under consideration are defined by duality for any  $w \in \mathcal{M}_\theta(\mathcal{S}^\beta)$  and  $v \in (\mathcal{S}^\beta)'$ . As proved in [25], the algebra  $C(\mathcal{S}^\beta)$  consists of the same elements as the space<sup>4</sup>  $(\mathcal{E}^\beta)'$ . We want to show that the topology of  $C(\mathcal{S}^\beta)$  is not weaker than  $\sigma((\mathcal{E}^\beta)', \mathcal{E}^\beta)$  and hence  $C'(\mathcal{S}^\beta) \supset ((\mathcal{E}^\beta)')' = \mathcal{E}^\beta$ .

If a functional  $u \in (\mathcal{S}^\beta)'$  belongs to  $C(\mathcal{S}^\beta)$ , then its continuous extension to  $\mathcal{E}^\beta$  can be constructed as follows. Let  $f_0$  be a function in  $\mathcal{S}^\beta$  with the property that  $\int f_0(\xi) d\xi = 1$ , and let  $h \in \mathcal{E}^\beta$ . We set

$$\langle u, h \rangle = \int \langle u, h(\cdot) f_0(\xi - \cdot) \rangle d\xi = \int (u * h_\xi)(\xi) d\xi, \quad \text{where } h_\xi(x) \stackrel{\text{def}}{=} h(\xi - x) f_0(x). \quad (28)$$

(the asterisk  $*$  here denotes ordinary convolution). The integrand in (28) is a continuous function because translations act continuously on  $\mathcal{S}^\beta$  and  $h$  is a pointwise multiplier of this space. There exists an  $N_0$  such that  $\|h\|_{-N_0, B} < \infty$  for each  $B > 0$ , and for the function  $h_\xi \in \mathcal{S}^\beta$ , we hence have the estimate

$$\begin{aligned} |\partial^\kappa h_\xi(x)| &\leq \sum_{\mu} \binom{\kappa}{\mu} |\partial^\mu h(\xi - x) \partial^{\kappa - \mu} f_0(x)| \\ &\leq \|h\|_{-N_0, B} \|f_0\|_{N, B} \sum_{\mu} \binom{\kappa}{\mu} B^{|\mu|} \mu^{\beta \mu} B^{|\kappa - \mu|} (\kappa - \mu)^{\beta(\kappa - \mu)} (1 + |\xi - x|)^{N_0} (1 + |x|)^{-N} \\ &\leq \|h\|_{-N_0, B} \|f_0\|_{N, B} (2B)^{|\kappa|} \kappa^{\beta \kappa} (1 + |x|)^{N_0 - N} (1 + |\xi|)^{N_0}. \end{aligned} \quad (29)$$

<sup>4</sup>In [25], the notation  $S^{\beta-}$  and  $\hat{\mathcal{E}}^{\beta-}$  was used instead of  $\mathcal{S}^\beta$  and  $\mathcal{E}^\beta$ . The space of pointwise multipliers for  $\mathcal{S}^\beta$  is shown there to be  $\text{proj} \lim_{B \rightarrow 0} \text{inj} \lim_{N \rightarrow \infty} E_N^{\beta, B}$ .

Therefore, we find that for any  $N, B > 0$ ,

$$\|h_\xi\|_{N,B} \leq \|h\|_{-N_0,B/2} \|f_0\|_{N+N_0,B/2} (1 + |\xi|)^{N_0}. \quad (30)$$

Because  $u \in C(\mathcal{S}^\beta)$ , for the neighborhood  $U = \{g: \|g\|_{N_0+d+1,1} < 1\}$  in  $\mathcal{S}^\beta$ , there exists a neighborhood  $V = \{g: \|g\|_{N,B} \leq \delta\}$  such that  $u * g \in U$  for all  $g \in V$ . We let  $\lambda_\xi$  denote the right-hand side of inequality (30). Then  $\delta \lambda_\xi^{-1} h_\xi \in V$ , and consequently

$$|(u * h_\xi)(\xi)| \leq \delta^{-1} \lambda_\xi (1 + |\xi|)^{-N_0-d-1} = \delta^{-1} \|h\|_{-N_0,B/2} \|f_0\|_{N+N_0,B/2} (1 + |\xi|)^{-d-1}. \quad (31)$$

Thus, the integral in (28) converges absolutely and the functional  $u$  is well defined on  $\mathcal{E}^\beta$ . Since the right-hand side of (31) contains the factor  $\|h\|_{-N_0,B/2}$ , this functional is continuous on every space  $\text{proj} \lim_{B \rightarrow 0} \mathcal{E}_N^{\beta,B}$ , and is hence continuous on  $\mathcal{E}^\beta$ . We must

also show that the functional defined by (28) when restricted to  $\mathcal{S}^\beta$  coincides with the original one, which justifies using the same symbol for them. If  $h \in \mathcal{S}^\beta$ , then an estimate similar to (29) yields the inequality

$$\|h(x) f_0(\xi - x)\|_{N,B} \leq \|h\|_{N+N_0,B/2} \|f_0\|_{N_0,B/2} (1 + |\xi|)^{-N_0}, \quad (32)$$

which holds for any  $N_0, N$ , and  $B$ . It follows that the integral in (28) remains absolutely convergent when  $u$  is replaced with any functional  $v \in (\mathcal{S}^\beta)'$ , because  $\|v\|_{N,B} \leq \infty$  for some  $N$  and  $B$ . The corresponding sequence of integral sums is hence weakly fundamental in  $\mathcal{S}^\beta$ . But  $\mathcal{S}^\beta$ , being a Montel space, is complete with respect to the weak convergence and this convergence implies the strong convergence (see Sec. I.6.3 in [24]). Therefore, the sequence of integral sums converges in  $\mathcal{S}^\beta$ , and its limit inevitably equals  $h$ , because the topology of  $\mathcal{S}^\beta$  is stronger than the topology of simple convergence.

Now let  $u_\gamma \rightarrow 0$  in  $C(\mathcal{S}^\beta)$ . Then for the bounded set

$$Q = \bigcap_{N,B} \{g: \|g\|_{N,B} \leq \|h\|_{-N_0,B/2} \|f_0\|_{N+N_0,B/2}\}$$

and for the neighborhood  $\epsilon U = \{g: \|g\|_{N_0+d+1,1} < \epsilon\}$  with arbitrarily small  $\epsilon$ , we can find  $\gamma_0$  such that  $u_\gamma * g \in \epsilon U$  for all  $g \in Q$  and all  $\gamma > \gamma_0$ . By (30) the family of functions  $(1 + |\xi|)^{-N_0} h_\xi$  is in  $Q$  and we conclude that

$$|\langle u_\gamma, h \rangle| \leq \int |(u_\gamma * h_\xi)(\xi)| d\xi \leq \epsilon \int (1 + |\xi|)^{-d-1} d\xi \quad \text{for all } \gamma > \gamma_0. \quad (33)$$

Therefore,  $\langle u_\gamma, h \rangle \rightarrow 0$  for any  $h \in \mathcal{E}^\beta$ , which completes the proof.  $\square$

The Gel'fand-Shilov spaces  $S_\alpha^\beta$  are DFN spaces, and the algebras  $\mathcal{M}_\theta(S_\alpha^\beta)$  were constructed and studied in [26]. A special role of the minimal nontrivial Fourier-invariant space  $S_{1/2}^{1/2}$  was also shown there. The other often used Gel'fand-Shilov spaces  $S^\beta = \text{inj} \lim_{B \rightarrow \infty} \text{proj} \lim_{N \rightarrow \infty} S_N^{\beta,B}$  are neither Fréchet nor DF spaces. This scale of spaces contains  $S^0$ , which is the Fourier transform of the space  $\mathcal{D}$  of smooth functions with compact support. The possibility of using  $S^0$  in noncommutative field theory was discussed in [18], [39]. We note that the spaces  $S^\beta$  with  $\beta > 0$  were not given a topology in [24], where the notion of convergence of sequences was used instead. The above consideration nevertheless applies to them because these spaces when equipped with a natural topology are nuclear [40], complete [41] and are obviously barrelled, being inductive limits of Fréchet spaces. As shown in [25], the algebra  $C(S^\beta)$  consists of the same elements as the dual of  $E^\beta = \text{inj} \lim_{N,B \rightarrow \infty} E_N^{\beta,B} = M(S^\beta)$ .

**Theorem 5.** *The space  $E^\beta$  is contained in  $\mathcal{M}_\theta(S^\beta)$ . For each  $w \in E^\beta$  and each  $v \in (S^\beta)'$ , the Moyal products  $w \star_\theta v$  and  $v \star_\theta w$  are well defined as elements of  $(S^\beta)'$ .*

*Proof.* Similarly to the above case of  $\mathcal{S}^\beta$ , it suffices to show that the topology of  $C(S^\beta)$  is not weaker than  $\sigma((E^\beta)', E^\beta)$ . A continuous extension of  $u \in (S^\beta)'$  to  $E^\beta$  can be defined by the same formula (28), but with  $f_0 \in S^\beta$ . This time  $\|h\|_{-N_0, B_0} < \infty$  for some  $N_0, B_0 > 0$  and an estimate analogous to (29) gives

$$\|h_\xi\|_{N, B_0} \leq \|h\|_{-N_0, B_0/2} \|f_0\|_{N+N_0, B_0/2} (1 + |\xi|)^{N_0} \quad \text{for any } N > 0. \quad (34)$$

Therefore, the family of functions  $(1 + |\xi|)^{-N_0} h_\xi$  is bounded in  $S^\beta$ . Let  $U$  be the neighborhood of the origin in  $S^\beta$  defined as the absolute convex hull of the set

$$\bigcup_B \{g: \|g\|_{N_0+d+1, B} < 1\}.$$

All functions in  $U$  are then dominated by  $(1 + |x|)^{-N_0-d-1}$ . Because the image of any bounded set under the continuous map  $g \rightarrow u * g$  is absorbed by the neighborhood  $U$ , we see that the integral in (28) converges and determines a continuous linear functional on  $E^\beta$ . If  $h \in S^\beta$ , then in this case, (32) holds for some  $B$  and for any  $N$  and  $N_0$ . Hence, the integral in (28) remains absolutely convergent when  $u$  is replaced with any linear functional defined and continuous on the Montel space  $\text{projlim}_{N \rightarrow \infty, \varepsilon \rightarrow 0} S_N^{\beta, B_1+\varepsilon}$ , where

$B_1 > B$ , and we deduce that the constructed functional restricted to  $S^\beta$  coincides with the initial one. Finally, let  $u_\gamma \rightarrow 0$  in  $C(S^\beta)$ . Then for the bounded set

$$Q = \bigcap_N \{g: \|g\|_{N, B_0} \leq \|h\|_{-N_0, B_0/2} \|f_0\|_{N+N_0, B_0/2}\}$$

and the neighborhood  $\epsilon U$ , there exists  $\gamma_0$  such that  $u_\gamma * g \in \epsilon U$  for all  $g \in Q$  and all  $\gamma > \gamma_0$ . By (34), the family  $(1 + |\xi|)^{-N_0} h_\xi$  is contained in  $Q$ , and we arrive at (33), which completes the proof.  $\square$

We note that for any  $\theta$ , the algebras  $\mathcal{M}_\theta(\mathcal{S}^\beta)$  and  $\mathcal{M}_\theta(S^\beta)$  contain all polynomials, as does the Moyal multiplier algebra of the Schwartz space. Furthermore, for any  $\beta \geq \beta_0$ , where  $0 < \beta_0 < 1$ , these algebras contain the entire functions of order  $\leq 1/(1 - \beta_0)$  (and minimal type in the case of  $\mathcal{M}_\theta(\mathcal{S}^{\beta_0})$ ), that are polynomially bounded on the real space. In particular, for any  $\beta \geq 0$ , the algebra  $\mathcal{M}_\theta(S^\beta)$  contains the space  $\mathcal{O}_{\text{exp}}$  consisting of entire functions of exponential type satisfying the condition  $|f(z)| \leq C(1 + |z|)^N e^{b|\text{Im } z|}$ , where the constants  $C$ ,  $N$  and  $b$  depend of  $f$ . This generalizes an earlier result [8], [10] for  $\mathcal{M}_\theta(S)$ .

*Remark 4.* Using reasoning similar to that for  $S_\alpha^\beta$  in the proof of Lemma 1 in [26], we can show that  $\mathcal{S}^\beta$  is dense in  $\mathcal{M}_\theta(\mathcal{S}^\beta)$ . In accordance with what was said at the end of Section 3, it follows that  $\mathcal{M}_\theta(\mathcal{S}^\beta)$  is an algebra under the Moyal product and  $(\mathcal{S}^\beta)'$  is an  $\mathcal{M}_\theta(\mathcal{S}^\beta)$ -bimodule. Analogously,  $(S^\beta)'$  is a bimodule over the algebra  $\mathcal{M}_\theta(S^\beta)$ .

The Weyl transforms of the constructed Moyal multiplier algebras, or, in other words, their operator realization in a Hilbert space, will be considered in a forthcoming paper.

## APPENDIX

The following simple lemma is useful in considering linear representations of topological groups in Montel function spaces.

**Lemma 1.** *Let  $E$  be a Montel space of functions on a set  $X$ , and let the topology of  $E$  is stronger than that of pointwise convergence. Let  $\mathcal{T}$  be a linear representation of a locally compact group  $\mathcal{G}$  in  $E$ . If for any function  $f \in E$ , there exists a compact neighborhood  $\mathcal{K}$  of unity  $e$  in  $\mathcal{G}$  such that the set  $\{\mathcal{T}_a f : a \in \mathcal{K}\}$  is bounded in  $E$  and if  $\lim_{a \rightarrow e} (\mathcal{T}_a f)(x) = f(x)$  for all  $x \in X$ , then the representation  $\mathcal{T}$  is continuous.*

*Proof.* If  $\{a_n\}$  is a sequence converging to  $e$  in  $\mathcal{G}$ , then  $a_n \in \mathcal{K}$  for sufficiently large  $n$ , and the sequence  $\{\mathcal{T}_{a_n} f\}$  is hence bounded in  $E$ . By the definition of a Montel space [27], such a sequence has at least one limit point. Since the topology of  $E$  is stronger than the topology of pointwise convergence, it follows from the limit relation  $\lim_{a \rightarrow e} (\mathcal{T}_a f)(x) = f(x)$  that only  $f$  can be its limit point, and therefore  $\mathcal{T}_{a_n} f \rightarrow f$ . Next, we use the generalized Banach-Steinhaus theorem (Sec. III.4.2 in [27]), which is applicable to Montel spaces. By that theorem, the pointwise boundedness of the operator set  $\{\mathcal{T}_a\}_{a \in \mathcal{K}}$  implies that for any absolutely convex neighborhood  $U$  of zero in  $E$ , there exists a neighborhood  $V$  such that  $\mathcal{T}_a(V) \subset U$  for all  $a \in \mathcal{K}$ . Writing  $\mathcal{T}_a f - f_0 = \mathcal{T}_a(f - f_0) + \mathcal{T}_a f_0 - f_0$ , we see that for the neighborhood  $U$  and any function  $f_0 \in E$ , there exists a neighborhood  $\mathcal{K}'(e)$  such that  $\mathcal{T}_a f \in f_0 + U$  for all  $f \in f_0 + \frac{1}{2}V$  and  $a \in \mathcal{K}'$ , i.e. the map  $\mathcal{G} \times E \rightarrow E : (a, f) \rightarrow \mathcal{T}_a f$  is continuous at the point  $(e, f_0)$ . Writing  $\mathcal{T}_a f - \mathcal{T}_{a_0} f_0 = \mathcal{T}_{a_0}(\mathcal{T}_{a_0^{-1}a} f - f_0)$ , we conclude that this map is continuous everywhere. The lemma is proved.  $\square$

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